

SOLUTION OF THE $SL(2,R)$ STRING IN CURVED SPACETIME¹

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ABSTRACT

The $SL(2,R)$ WZW model, one of the simplest models for strings propagating in curved space time, was believed to be non-unitary in the algebraic treatment involving affine current algebra. It is shown that this was an error that resulted from neglecting a zero mode that must be included to describe the correct physics of non-compact WZW models. In the presence of the zero mode the mass-shell condition is altered and unitarity is restored. The correct currents, including the zero mode, have logarithmic cuts on the worldsheet. This has physical consequences for the spectrum because a combination of zero modes must be quantized in order to impose periodic boundary conditions on mass shell in the physical sector of the theory. To arrive at these results and to solve the model completely, the $SL(2,R)$ WZW model is quantized in a new free field formalism that differs from previous ones in that the fields and the currents are Hermitean, there are cuts, and there is a new term that could be present more generally, but is excluded in the WZW model.

1 Introduction

One of the main reasons to study string theory in curved spacetime is to develop the appropriate methods to investigate physical phenomena in the presence of quantum gravity in a mathematically consistent theory. Quantum gravity is important during the early universe and this must have an impact on the symmetries and matter content (gauge bosons, families of quarks and leptons) observed at accelerator energies. In addition, in order to develop an understanding of gravitational singularities such as black holes or the big bang, including quantum aspects of gravity, string theory in curved spacetime must be investigated.

A string propagating in curved spacetime with one time and $(d-1)$ space coordinates is described by a string action that has the following form in the conformal gauge

$$S = \int d\tau d\sigma [\partial_+ X^\mu \partial_- X^\nu G_{\mu\nu}(X) + \cdots] \quad (1)$$

¹Based on lectures delivered at the Strings '95 conference, USC, March 1995, and at the Strings, Gravity and Physics at the Planck Scale conference, Erice, August 1995.

where $G_{\mu\nu}(X)$ is a background metric in d -dimensions with signature $(-1, 1, 1, \dots)$. The terms in the action denoted by \dots may contain additional background fields such as an antisymmetric tensor $B_{\mu\nu}(X)$ a dilaton $\Phi(X)$ etc.. Conformal invariance must be imposed on these background fields at the quantum level, otherwise reparametrization invariance (Virasoro constraints) cannot be used to remove ghosts from the theory.

In one time plus $(d - 1)$ space dimensions many models that are exactly conformally invariant at the quantum level have been constructed by now [1][2][3]. A first example of string propagation in curved spacetime was the $SL(2, R)$ WZW model that potentially could be solved through algebraic methods. However, one immediately came across an unexpected inconsistency problem involving the unitarity of the theory [1][4]. It was noticed a long time ago that the Virasoro constraints were insufficient to remove all the negative norm states encountered in the affine current algebra treatment of the model. There seemed to be more negative norm states than those introduced by the time-like string coordinates. This problem persisted with all other non-compact affine current algebra coset models, including the much studied $SL(2, R)/R$ two dimensional black hole [1][5]. This observation casted doubt on the physical validity of the models and prevented the development of physical ideas based on the algebraic properties of these (in principle) completely solvable models.

In this talk, and in a related paper [6], I explain the solution of the unitarity problem. I show that the WZW model has more degrees of freedom than those described by the affine current algebra. The extra degrees of freedom are zero modes (in particular, in non-compact directions). In the old treatment one inadvertently set the zero mode to zero values, and hence missed important properties of the model. One of the important effects of the non-trivial zero mode is that the mass shell condition is altered. In the absence of the zero mode the string cannot be put correctly on mass shell and this begins to explain why the non-unitary states emerged.

To actually show that the model has no ghosts a second step is needed. In particular one must argue that the discrete series representation of $SL(2, R)$ does not appear at the base, since in the purely algebraic approach (including the new zero mode) there would still be negative norm states in this representation, according to the old approach. To show that the WZW model excludes the discrete series, I formulate the quantum theory in terms of a suitable parametrization of $SL(2, R)$ that corresponds to free fields, and then show that only the unitary principal series is allowed. In this formulation the spectrum is completely solved and the Virasoro constraints implemented. The no ghost theorem is then shown to be valid.

In the presence of non trivial values of the zero mode (which is needed for the correct physics) the conserved currents of the theory have logarithmic singularities on the world sheet. *A priori* one may think that this would imply that the closed string boundary conditions are not satisfied. However, this is not the case, because periodicity is required only on mass shell. In the presence of the new zero modes the

Hilbert space is larger than before. Imposing periodicity gives quantized values for a combination of the zero modes, thus satisfying the correct closed (or open) string boundary conditions for the physical on-mass-shell states.

To arrive at these results, and also to solve the model completely, a new free field formalism is introduced as mentioned above. The structure of the currents in terms of the free fields is reminiscent of the one found by Wakimoto [7] and Gerasimov et. al. [8], but it differs from theirs in three important aspects: (1) The free fields as well as the currents are Hermitean; this is important for the discussion of unitarity. (2) The currents have logarithmic cuts that are associated naturally with the zero modes of the free fields; this affects the spectrum and monodromy. (3) A new term, that can be present in the free field formulation to reproduce the effects of the most general $SL(2, R)$ currents, is introduced in order to obtain all unitary representations at the base. The source of all the unwanted ghosts is traced to the new term. But, for the specific model at hand, i.e. the WZW model, the extra term is shown to be absent, thus elucidating the mechanism by which the model becomes unitary.

2 The unitarity problem

The $SL(2, R)$ WZW model has a timelike coordinate that introduces negative norm oscillators. These create negative norm states, but this is not the source of the problem. On the basis of naïve counting one may hope that the Virasoro constraints will remove these ghosts from the theory (this expectation is born out in our final result). A similar situation occurs also in the flat theory. As is well known, in the flat case one can indeed prove the no ghost theorem [9] which implies that the theory is unitary. However, in the case of curved spacetime current algebra models, one finds that even after imposing the Virasoro constraints there remains negative norm states that render the theory non-unitary. This has been the main stumbling block that discouraged the application of these ideas to model building for the past five years.

Negative norm states that satisfy the Virasoro constraints can be displayed explicitly. An example is [1]

$$\begin{aligned} |\phi, l\rangle &= \left(J_{-1}^1 - iJ_{-1}^2\right)^l |j, m = j + 1\rangle, \\ \tilde{L}_n |\phi, l\rangle &= 0, \quad n \geq 1, \\ \langle \phi, l | \phi, l \rangle &= N_{j(l)} (l!) \prod_{r=0}^{l-1} (k - 2j(l) - 2 + r). \end{aligned} \tag{2}$$

where $N_{j(l)} = \langle j, m = j + 1 | j, m = j + 1 \rangle$ is the norm of the state at the base. The base is in the discrete series representation of $SL(2, R)$, with $m \geq (j + 1)$. It is required to be in the discrete series^a by the mass-shell condition $L_0 = 1$, which gives

$$-\frac{j(j+1)}{k-2} + l = 1. \tag{3}$$

^aWhen $j(j+1) > 0$ only the discrete series can occur among the unitary representations of $SL(2, R)$. By contrast, the principal series occurs only when $j(j+1) < -1/4$.

Evidently, for sufficiently large values of the excitation number l the norm switches between positive and negative values. Hence, despite the Virasoro constraints this model is not unitary and cannot describe a physical string.

3 Solution of the unitarity problem

Until now a solution to this problem, and the related $SL(2, \mathbb{R})/\mathbb{R}$ black hole problem, has not been found despite many attempts [10][11]. Suggestions included: (1) Restrict (artificially) $j(l) + 1 < k/2$ so that the norm never becomes negative; (2) Allow large values of $j(l)$ as needed by the excited level l , but also permit the base state to have negative norm $N_{j(l)}$ in such a way as to make the norm of the excited state $< \phi, l | \phi, l >$ positive; (3) Hope that modular invariants will fix the problem. All of these suggestions are rejected [6].

The resolution of the problem lies in understanding that wrong assumptions have been made about the algebraic structure of the WZW model. In particular, the assumption that the $SL(2, \mathbb{R})$ WZW model is described by affine $SL(2, \mathbb{R})$ is not entirely correct. There is an additional zero mode that is present in the local conserved $SL(2, \mathbb{R})$ currents of the WZW model, whose presence is crucial for the resolution of the unitarity issues. This zero mode is missed by the assumption of affine currents that are written in the form of a Laurent series $\tilde{J}^a(z) = \sum J_n^a z^{-n-1}$. When the additional zero mode is included, the true currents $J^a(z)$ have a logarithmic term $\ln z$ in addition to the usual powers z^n . Hence, manipulations such as (2,3) based on the old affine currents $\tilde{J}^a(z)$ do not fully reflect the correct theory, and this is why we find inadmissible unphysical results.

In general one can include such $\ln z$ parts, but still have the correct local commutation rules or operator products with only poles. To see this, let $\tilde{J}^a(z)$ be the usual Laurent series (with the usual operator products) that have modes J_n^a as above, and in addition introduce a new zero mode α_0^- that commutes with all the other modes J_n^a . The new currents are

$$\begin{aligned} J^0(z) + J^1(z) &= [\tilde{J}^0(z) + \tilde{J}^1(z)] \\ J^0(z) - J^1(z) &= [\tilde{J}^0(z) - \tilde{J}^1(z)] - 2i\alpha_0^- \ln z \tilde{J}^2(z) \\ &\quad - \frac{k}{z}\alpha_0^- + (-i\alpha_0^- \ln z)^2 [\tilde{J}^0(z) + \tilde{J}^1(z)] \\ J^2(z) &= \tilde{J}^2(z) - i\alpha_0^- \ln z [\tilde{J}^0(z) + \tilde{J}^1(z)] \end{aligned} \tag{4}$$

It can be shown that they have the usual correct operator products, with only poles, for any value of the zero mode α_0^- [6].

The $\ln z$ terms arise naturally in the canonical formulation of the WZW model. There are left/right moving string coordinates $X_{L,R}$ that parametrize the group element $g(X_L(z), X_R(\bar{z}))$. As usual, string coordinates have a “momentum” zero mode $p \ln z$. The currents in this model depend both on $X_{L,R}$ as well as on their derivatives.

Therefore, the currents are expected to include $\ln z$ pieces proportional to the zero modes. This is unlike the flat theory, where the currents depend only on the derivatives of the $X_{L,R}$. If one sets the $p \ln z$ parts equal to zero, as is inadvertently done by assuming affine currents in the form of Laurent series, one forces the string to lie in a sector of fixed zero mode $p = 0$. This may be harmless in compact directions, but it is fatal in non-compact directions. For example, for the flat string, if one requires the lightcone momentum $p^- = 0$, then the mass-shell condition

$$p^+ p^- - p_i^2 = l - 1$$

cannot be satisfied in that sector. In the $SL(2,R)$ string exactly this situation arises when $\alpha_0^- = 0$. It turns out that, in $SL(2,R)$, in the absence of the zero mode, which is analogous to p^- , it is still possible to satisfy a mass shell condition, but only in the discrete series representation. As seen above the discrete series gives rise to ghosts. On the other hand, when the zero mode $\alpha_0^- \neq 0$ is included (and hence $\ln z$ is present in the currents), the mass shell condition is satisfied in the principal series representation, which is free of ghosts at the excited levels.

4 $SL(2,R)$ Currents and free fields

Consider the free fields $X^-(z)$, $P^+(z)$, $S(z)$, $T'(z)$. They have naïve dimensions 0,1,1,2 respectively and they are defined as follows

$$\begin{aligned} X^-(z) &= q^- - i\alpha_0^- \ln z + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^- z^{-n}, \quad (\alpha_n^-)^\dagger = \alpha_{-n}^- \\ P^+(z) &= \sum_{n=-\infty}^{\infty} \alpha_n^+ z^{-n-1}, \quad (\alpha_n^+)^\dagger = \alpha_{-n}^+ \\ S(z) &= \sum_{n=-\infty}^{\infty} s_n z^{-n-1}, \quad (s_n)^\dagger = s_{-n} \\ T'(z) &= \sum_{n=-\infty}^{\infty} L'_n z^{-n-2}, \quad (L'_n)^\dagger = L'_{-n} \end{aligned} \tag{5}$$

These fields are Hermitean. The currents are

$$\begin{aligned} J_0(z) + J_1(z) &= P^+(z) \\ J_0(z) - J_1(z) &= :X^-(z) P^+(z) X^-(z): + 2S(z) X^-(z) \\ &\quad - ik \partial_z X^-(z) - \frac{(k-2)T'(z)}{P^+(z)} \\ J_2(z) &= :X^-(z) P^+(z): + S(z) \end{aligned} \tag{6}$$

The currents have a Wakimoto type structure, but with the exception that these currents are Hermitean. Furthermore, they contain two other new features: (i) the

In z parts that have the same structure as (4), and (ii) the new terms T'/P^+ . It can be shown that the operator products of free fields produce the correct operator products of the currents [6].

The free field commutation rules are

$$\begin{aligned} [q^-, \alpha_0^+] &= i, \\ [\alpha_n^-, \alpha_m^+] &= n \delta_{n+m,0}, \\ [s_n, s_m] &= \left(\frac{k}{2} - 1\right) n \delta_{n+m,0}, \\ [L'_n, L'_m] &= (n - m) L'_{n+m} + 0 \end{aligned} \tag{7}$$

while all other commutators are zero. The α_n^\pm oscillators may be rewritten in terms of light-cone type combinations of one time-like α_n^0 and one space-like α_n^1 oscillator, i.e. $\alpha_n^\pm = (\alpha_n^1 \pm \alpha_n^0)/\sqrt{2}$. This shows one source of negative norms, but by naive counting, they are expected to be removed by the Virasoro constraints. In fact, the usual proof of the no-ghost theorem can be extended to show that these ghosts are indeed removed [6]. The zero modes q^-, p^\pm are interpreted as light-cone type canonical variables $q = x^-, p = p^+$. We have not introduced a canonical variable corresponding to x^+ , hence $\alpha_0^- = p^-$ commutes with all the operators and acts like a constant. Similarly the zero mode s_0 also acts like a constant. α_0^\pm, s_0, L'_0 are simultaneously diagonalized in the Hilbert space, and they label the base.

The L'_n operators act like Virasoro operators with zero central charge. It is always possible to construct such an operator in terms of free fields, the simplest being a (negative norm) free boson with a background charge. Another example is any critical conformal field theory including the conformal ghosts with central charge $c = -26$. Both of these examples have ghosts and this is true more generally. Indeed, there is no construction of a zero central charge Virasoro operator that does not contain negative norm states in its Hilbert space. Hence, in the presence of L'_n there is an additional source of negative norms that will not be possible to be removed by the Virasoro constraints. For example a base state with the property $L'_0|h' \rangle = h'|h' \rangle$ produces a negative norm state $|\phi \rangle = L'_{-1}|h' \rangle$, $\langle \phi|\phi \rangle = 2h' < h'|h' \rangle$, if $h' < 0$. As we will see below only when L' is present and $h' < 0$, the discrete series is possible. Hence the unwanted negative norm states and the discrete series go hand-in-hand. We will show that L' is absent in the WZW model, hence the WZW model has no additional sources of negative norms.

5 Stress tensor and free fields

The energy momentum tensor is obtained from the normal ordered product of the currents

$$T(z) = \frac{1}{k-2} : \left(-(J_0(z))^2 + (J_1(z))^2 + (J_2(z))^2 \right) : \tag{8}$$

The result of the computation gives

$$T(z) = : P^+ i \partial_z X^- : + T_S(z) + T'(z) \quad , \quad (9)$$

If the computation is repeated with the $\tilde{J}(z)$ currents the only difference is dropping the α_0^- term contained in

$$i \partial_z X^- = \sum_{n=-\infty}^{\infty} \alpha_n^- z^{-n-1}. \quad (10)$$

In (9) T_S is a Hermitean stress tensor

$$T_S(z) = \frac{1}{k-2} \left[: (S(z))^2 : - \frac{i}{z} \partial_z (z S(z)) + \frac{1}{4z^2} \right] \quad (11)$$

The structure $\frac{i}{z} \partial_z (z S(z))$ differs from the usual one $i \partial S$, and thus is Hermitean. The operator products of $T_S(z)$ are

$$T_S(z) \times T_S(w) = \frac{c_s/2}{(z-w)^2} + \frac{2T_S(w)}{(z-w)^2} + \frac{\partial_w T_S(w)}{z-w} + \dots \quad (12)$$

with the central charge

$$c_s = 1 + \frac{6}{k-2}. \quad (13)$$

Note that the term $P^+ i \partial_z X^-$ is identical to the energy momentum tensor of flat light-cone coordinates constructed from the oscillators α_n^\pm . Therefore, that part is mathematically equivalent to a $c = 2$ stress tensor constructed from one time and one space coordinate in flat spacetime. Then the total central charge is

$$\begin{aligned} c &= 2 + c_s + c' \\ &= 2 + \left(1 + \frac{6}{k-2}\right) + 0 \\ &= \frac{3k}{k-2}, \end{aligned} \quad (14)$$

which is the right central charge for the $SL(2, R)$ WZW model. Finally, as a further consistency check, by using only the operator products of the elementary fields, one finds that $T(z)$ has the correct operator products with the currents.

The zero mode of the stress tensor takes the form $L_0 = L_0^\pm + L_0^S + L_0'$, where each piece has the eigenvalues

$$\begin{aligned} L_0^\pm &= p^+ p^- + l_\pm, \\ L_0^S &= (s_0^2 + 1/4)/(k-2) + l_s, \\ L_0' &= h' + l'. \end{aligned} \quad (15)$$

where l_\pm, l_s, l' are positive integers and h' is the eigenvalue of L_0' at the base (whose possible values depend on the model for T'). The mass shell condition $L_0 = a$ is

$$p^+ p^- + (s_0^2 + 1/4)/(k-2) + h' + \text{integer} = a \quad (16)$$

where $a \leq 1$. The term p^+p^- is crucial since it takes negative values, as seen below.

To identify the value of the Casimir operator $j(j+1)$ associated with the affine currents \tilde{J} in (4) we set $p^- = 0$, and compare the eigenvalue of the resulting L_0 to the standard formula. We then see that the Casimir of the old currents $(\tilde{J}_0)^2 = -j(j+1)$ takes the value

$$j(j+1) = -(s_0^2 + 1/4) - h'(k-2). \quad (17)$$

Therefore, if the T' piece is absent in the construction ($h' = 0$), then $j = -1/2 \pm is_0$ is only in the principal series. The supplementary series could occur for $-1/4 < j(j+1) < 0$ and the discrete series occurs for $-1/4 < j(j+1)$. We see that the field T' with a positive h' contributes only to the principal series and with a negative h' it leads to the other representations as well. This construction may find various applications in the future. We will see below that T' is absent in the $SL(2, \mathbb{R})$ WZW model, hence only the special case of our construction ($T' = 0$) finds an application in the WZW model. Then, for excited string states, since the integer in (16) is positive it would not be possible to satisfy the mass shell condition in the absence of the p^- . So, the logarithmic structure plays a role.

6 The $SL(2, \mathbb{R})$ WZW model

We now relate the algebraic structures above to the WZW model for $SL(2, \mathbb{R})$.

The quantum theory for any WZW model at the critical point is conveniently formulated in terms of the left and right moving currents after writing the group element $g(\tau, \sigma) = g_L(\tau + \sigma) g_R^{-1}(\tau - \sigma)$

$$J_L(z) = ik\partial_z g_L g_L^{-1}, \quad J_R(\bar{z}) = ik\partial_{\bar{z}} g_R g_R^{-1}, \quad (18)$$

where $z = e^{i(\tau+\sigma)}$, $\bar{z} = e^{i(\tau-\sigma)}$. The quantum rules are most conveniently given in terms of operator products among the currents and the group elements.

$$\begin{aligned} J_{L,R}^i(z) J_{L,R}^j(w) &\rightarrow \frac{k/2}{(z-w)^2} + i\epsilon^{ijl} \eta_{lk} \frac{J_{L,R}^k(w)}{z-w} + \dots \\ J_{L,R}^i(z) g_{L,R}(w) &\rightarrow \frac{-t^i}{z-w} g_{L,R}(w) + \dots \end{aligned} \quad (19)$$

The t_i is a basis for the $SL(2, \mathbb{R})$ Lie algebra which is given in terms of Pauli matrices $t_0 = \sigma_2/2$, $t_1 = i\sigma_1/2$, $t_2 = -i\sigma_3/2$. They satisfy $\eta_{ij} = -2\text{tr}(t_i t_j) = \text{diag}(-1, 1, 1)$.

Any group element $g_{L,R}$ in $SL(2, \mathbb{R})$ can be rewritten in terms of the Gauss de-

composition as follows^b

$$g_{L,R} = \begin{pmatrix} 1 & 0 \\ X_{L,R}^- & 1 \end{pmatrix} \begin{pmatrix} : e^{-\frac{u_{L,R}}{k-2}} : & 0 \\ 0 & : e^{\frac{u_{L,R}}{k-2}} : \end{pmatrix} \begin{pmatrix} 1 & X_{L,R}^+ \\ 0 & 1 \end{pmatrix} \quad (20)$$

We compute the left/right currents (omitting the L, R indices for simplicity)

$$i(k-2) : \partial_z g g^{-1} : = \begin{pmatrix} -J^2(z) & J^0(z) + J^1(z) \\ -J^0(z) + J^1(z) & J^2(z) \end{pmatrix} \quad (21)$$

As compared to the classical currents (18) we have shifted $k \rightarrow (k-2)$ in both g (20) and the definition of the current (21), and applied normal ordering. This renormalization is necessary for the commutation rules to work out, and is consistent with similar phenomena concerning the quantization of the WZW model [16]. One finds then

$$\begin{aligned} J^0(z) + J^1(z) &= : (k-2) i \partial_z X^+ e^{-2u/(k-2)} : \\ J^2(z) &= (k-2) : i \partial_z X^+ X^- e^{-2u/(k-2)} : + i \partial_z u \\ J^0(z) - J^1(z) &= (k-2) : i \partial_z X^+ (X^-)^2 e^{-2u/(k-2)} : \\ &\quad + 2X^- i \partial_z u - ik \partial_z X^- \end{aligned} \quad (22)$$

The coefficient of $-ik \partial_z X^-$ is ambiguous because of the normal ordering of the term $: i \partial_z X^+ (X^-)^2 e^{-2u/(k-2)} :$. Again this has to be fixed by requiring that the commutation rules work out. Therefore, instead of having naively $-i(k-2) \partial_z X^-$, we actually must have $-ik \partial_z X^-$. These results are established by applying the canonical formalism and identifying these structures with canonical conjugate variables. Velocities must be replaced by canonical momenta. Note that for left/right movers ∂_z can be related to time derivatives ∂_τ or space derivatives ∂_σ . So, at the quantum level we find that we must identify the canonical pairs (X^-, P^+) and (u, S) as follows

$$\begin{aligned} P^+(z) &= (k-2) i \partial_z X^+ e^{-2u/(k-2)} \\ S(z) &= i \partial_z u \end{aligned} \quad (23)$$

and then the currents take the form

$$\begin{aligned} J^0(z) + J^1(z) &= P^+(z) \\ J^2(z) &= : X^- P^+ : + S \\ J^0(z) - J^1(z) &= : X^- P^+ X^- : + 2S X^- - ik \partial_z X^- \end{aligned} \quad (24)$$

^bFor any $SL(2, \mathbb{R})$ group element $g = (a, b; c, d)$, the Gauss decomposition is given by $X^+ = b/a$, $X^- = c/a$. Instead of the exponentials in the middle factor $\exp[\mp u/(k-2)]$ one could take more generally $diag(a, a^{-1})$, where a can have any sign, unlike the exponentials. One may carry out the quantization in terms of a instead of u . However, in the final analysis the currents depend only on a^2 or on $S \sim a \partial a^{-1}$ which may be rewritten as $S \sim |a| \partial |a|^{-1}$ even if a changes sign. Therefore, the parametrization used here, $|a| = \exp[-u/(k-2)]$, is adequate for the general case.

This is the form used in the previous section without the extra field $L'(z)$. Thus, as discussed before, only the principal series will emerge in the WZW model. Using the oscillator form introduced in (5) we can express $u(z)$ and $X^+(z)$ in terms of the basic oscillators s_n, α_n^+ by inverting the formulas in (23), thus

$$\begin{aligned} u(z) &= u_0 - i s_0 \ln z + i \sum_{n \neq 0} \frac{1}{n} s_n z^{-n} \\ X^+(z) &= -i \int^z dz' \frac{P^+(z')}{(k-2)} : \exp \left[\frac{2u(z')}{k-2} \right] : . \end{aligned} \quad (25)$$

Then these structures satisfy the operator products

$$\begin{aligned} < u(z) S(w) > = \left(\frac{i}{z-w} + \frac{i}{2w} \right) \left(\frac{k}{2} - 1 \right) \\ [J^0(z) - J^1(z)] \times X^+(w) \rightarrow \frac{-i}{z-w} : e^{2u(w)/(k-2)} : \end{aligned} \quad (26)$$

Thus, $u(z)$ is just the canonical conjugate to $S(z)$. Another property of X^+ that follows from the fundamental operator products is that it is a singlet under the action of $J_2(z)$

$$J_2(z) \times X^+(w) \rightarrow 0. \quad (27)$$

Actually ∂X^+ is a screening current (see below). Its operator products with all the currents is either zero or a total derivative. Therefore, its zero mode commutes with all the currents.

Inserting the expressions in eq.(25) into (20) we obtain the quantum operator version of the group element g . The operator products may now be evaluated. We find the correct quantum products (19) with the above construction in terms of oscillators. That is,

$$\begin{aligned} [J_{L,R}^0(w) + J_{L,R}^1(w)] \times g_{L,R}(w) &\rightarrow \frac{-i}{z-w} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} g_{L,R}(w) \\ J_{L,R}^2(z) \times g_{L,R}(w) &\rightarrow \frac{i/2}{z-w} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g_{L,R}(w) \\ [J_{L,R}^0(w) - J_{L,R}^1(w)] \times g_{L,R}(w) &\rightarrow \frac{i}{z-w} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} g_{L,R}(w) \end{aligned} \quad (28)$$

This result, combined with the current \times current operator products that we have proven earlier, is convincing evidence that the free field formalism that we have discussed corresponds to the quantization of the $SL(2, \mathbb{R})$ WZW model.

7 Physical states

7.1 No ghosts

Since we have rewritten the WZW theory in terms of free fields, the space of states consists of the Fock space for the oscillators α_n^\pm, s_n applied on the base $|p^+, p^-, s_0 >$

that diagonalizes the zero mode operators α_0^\pm, s_0 .

$$\prod_{n=1}^{\infty} (\alpha_{-n}^+)^{a_n} \prod_{m=1}^{\infty} (\alpha_{-m}^-)^{b_m} \prod_{k=1}^{\infty} (s_{-k})^{c_k} |p^+, p^-, s_0 \rangle \quad (29)$$

where the powers a_n, b_m, c_k are positive integers or zero. This is the space of states that provide a representation basis for the $SL(2, \mathbb{R})$ currents with only the principal series. The physical states are identified as those linear combinations that are annihilated by the the total Virasoro generators

$$L_n |\psi \rangle = 0, \quad n \geq 1. \quad (30)$$

In the present case the total Virasoro generators include the following terms

$$L_n = L_n^\pm + L_n^S. \quad (31)$$

where

$$\begin{aligned} L_n^\pm &= \sum_m : \alpha_{-m}^\mp \alpha_{n+m}^\pm : \\ L_n^S &= \frac{1}{k-2} \left(\sum_m : s_{-m} s_{n+m} : + i n s_n + \frac{1}{4} \delta_{n,0} \right) \end{aligned} \quad (32)$$

Note that the L_n^\pm is equivalent to the $c = 2$ Virasoro operator in 2D flat spacetime. The full central charge is

$$c = \frac{3k}{k-2}. \quad (33)$$

The eigenvalue of the total L_0 is

$$L_0 = p^+ p^- + \frac{1}{k-2} [s_0^2 + 1/4] + \text{integer} = a \quad (34)$$

Thus, the theory has been reduced to a 2D lightcone in flat spacetime plus a Liouville type space-like free field that has positive norm. A small but important difference as compared to the standard Liouville formalism is that the linear term in L_n^S is Hermitean in our case, and does not contribute to L_0^S .

The only negative norm states are the ones produced by the time-like oscillator $\alpha_n^0 = (\alpha_n^+ - \alpha_n^-) / \sqrt{2}$. However, this is no worse than the usual flat spacetime case. The space of physical states is defined by

$$(L_n - a \delta_{n,0}) |\phi \rangle = 0 \quad (35)$$

with $a \leq 1$ fixed. A proof of no ghosts can now be given by following step by step the same arguments that prove the no ghost theorem in flat spacetime [9]. There is no need to repeat it here. We only recall that there are no ghosts as long as $a \leq 1$ and $c \leq 26$.

7.2 Monodromy

So far we have not taken into account the physical effects of the $\ln z$ cut in the currents. At first sight, the presence of $\ln z$ in the currents appears to be contrary to the periodicity requirement of closed strings. However, this is not true. The periodicity requirement arises as a boundary condition in the process of minimizing the action. In other words only on mass shell physical string configurations are required to be periodic. In the presence of the extra zero mode the Hilbert space is larger. One must require periodicity in the physical on shell sector. The physical sector is identified as the subspace of states for which the matrix elements of the currents are periodic.

$$\langle phys | J^i(z e^{i2\pi n}) | phys' \rangle = \langle phys | J^i(z) | phys' \rangle . \quad (36)$$

As described below, under this requirement, it turns out that the extra zero mode must have quantized eigenvalues in the physical sector. Quantum mechanically it is possible to impose the monodromy condition simultaneously with the Virasoro constraints since the latter commute with the monodromy operator as seen below. These monodromy conditions are easily taken care of in the free field formalism, thus finally giving a complete physical unitary spectrum of the $SL(2, \mathbb{R})$ WZW model.

To implement the monodromy let us first consider its effect on the currents. From the modified currents in (4) we see that under the monodromy the currents undergo a linear transformation

$$\begin{aligned} [J^0 + J^1](z e^{i2\pi n}) &= [J^0 + J^1](z) \\ [J^0 - J^1](z e^{i2\pi n}) &= [J^0 - J^1](z) + 4\pi n \alpha_0^- J^2(z) \\ &\quad + (2\pi n \alpha_0^-)^2 [J^0 + J^1](z) \\ J^2(z e^{i2\pi n}) &= J^2(z) + 2\pi n \alpha_0^- [J^0 + J^1](z) \end{aligned} \quad (37)$$

Note that on the right hand side one finds J^a , not \tilde{J}^a . Therefore we expect that the right hand side can be rewritten as the adjoint action with a global $SL(2, \mathbb{R})$ transformation. Since the current $J^0(z) + J^1(z)$ remains unchanged the generator of this transformation must be the zero mode of this current. Indeed, since α_0^- acts like a number, we can rewrite the monodromy in the form

$$J^i(z e^{i2\pi n}) = e^{-2i\pi n \alpha_0^- (J_0^0 + J_0^1)} J^i(z) e^{2i\pi n \alpha_0^- (J_0^0 + J_0^1)} \quad (38)$$

Therefore physical states that satisfy (36) are the subset of states that are invariant under the monodromy

$$e^{2i\pi n \alpha_0^- (J_0^0 + J_0^1)} | phys \rangle = | phys \rangle . \quad (39)$$

In the free boson representation this is easy to implement. Using $(J_0^0 + J_0^1) = \alpha_0^+$ this condition is applied on the Fock space of the free bosons in the form

$$e^{2i\pi n \alpha_0^- \alpha_0^+} \prod_{n,m,k=1}^{\infty} (\alpha_{-n}^+)^{a_n} (\alpha_{-m}^-)^{b_m} (s_{-k})^{c_k} | p^+ p^- s_0 \rangle \quad (40)$$

Since $\alpha_0^- \alpha_0^+$ commutes with all oscillators, it can be moved to the right and applied on the base. The result is the quantization condition

$$\begin{aligned} e^{2i\pi n \alpha_0^- \alpha_0^+} |p^+, p^-, s_0\rangle &= |p^+, p^-, s_0\rangle \\ \alpha_0^- \alpha_0^+ &= p^- p^+ = -r, \quad r = 0, 1, 2, \dots \end{aligned} \quad (41)$$

We must take negative integers because according to the mass shell condition $p^- p^+$ is negative. So, the mass shell condition on physical states at excitation level l takes the form

$$-r + \frac{1}{k-2} [s_0^2 + 1/4] + l = a. \quad (42)$$

It is always possible to satisfy this condition with some value of s_0 which is quantized in terms of the positive integers r, l . In terms of the original Casimir $j(j+1)$ this corresponds to a principal series representation of $SL(2, \mathbb{R})$ with quantized values of j given by $j = -\frac{1}{2} + is_0 = -\frac{1}{2} \pm i\sqrt{(k-2)(r-l+a) - 1/4}$ where r must be chosen so that the square root is real.

Therefore, it is sufficient to require that j (or s_0) has quantized values

$$j_n = -\frac{1}{2} \pm i\sqrt{(k-2)(n+a) - 1/4}. \quad (43)$$

Then the on-shell physical states automatically satisfy the periodicity condition.

7.3 Open and Closed strings

An open string action $S = \int d\tau \int_0^\pi d\sigma L(\tau, \sigma)$ is minimized by allowing free variation of the end points. For the WZW model for any group G this produces the boundary terms

$$\delta S = \int d\tau \left\{ \begin{array}{l} Tr((\delta g g^{-1})(\partial_\sigma g g^{-1}))|_\pi \\ - Tr((\delta g g^{-1})(\partial_\sigma g g^{-1}))|_0 \end{array} \right\} \quad (44)$$

In addition to the equations of motion, these terms must also vanish at each end of the string. That is,

$$\partial_\sigma g g^{-1}|_{\sigma=0} = 0 = \partial_\sigma g g^{-1}|_{\sigma=\pi}. \quad (45)$$

At the conformal critical point the equations of motion are satisfied by the general form $g(\tau, \sigma) = g_L(\tau + \sigma) g_R^{-1}(\tau - \sigma)$. Then the boundary conditions require that g_L and g_R be related to each other by the constraint

$$g_L^{-1}(\tau) \partial_\tau g_L(\tau) + g_R^{-1}(\tau) \partial_\tau g_R(\tau) = 0. \quad (46)$$

Furthermore, each term in this equation is required to be periodic. As discussed in the rest of this paper, we impose periodicity on the physical states. The relation (46) between $g_L(\tau)$ and $g_R(\tau)$ is not easy to solve explicitly. However, we may carry out the quantum theory in terms of one current \hat{J}

$$\hat{J}(z) = g_L^{-1}(z) \partial_z g_L(z) = -g_R^{-1}(z) \partial_z g_R(z). \quad (47)$$

This is neither the left moving current $J_L = \partial g_L g_L^{-1}$ nor the right moving one $J_R = \partial g_R g_R^{-1}$, but is related to them by transformations involving g_L or g_R .

$$J_L = g_L \hat{J} g_L^{-1}, \quad J_R = -g_R \hat{J} g_R^{-1}.$$

The current \hat{J} generates transformations on the right side of g_L and the left side of g_R^{-1} , and the meaning of (46) is that the total current on both g_L and g_R vanishes at the end points. The canonical commutation rules for this current are identical to the ones we have already discussed in the rest of the paper. The stress tensor constructed from it is equal to the stress tensor constructed from either the left movers or the right movers

$$Tr(\hat{J}^2) = Tr(J_L^2) = Tr(J_R^2). \quad (48)$$

The quantum spectrum is obtained from the properties of \hat{J} , whose mathematical structure is the same as either left movers or right movers as discussed in the previous sections. For an open string we choose to parametrize the \hat{J} current in terms of free fields. Thus, the quantum spectrum of the open string in the $SL(2, \mathbb{R})$ curved spacetime becomes identical to the spectrum discussed above.

For a closed string we have independent left and right moving sectors. The full group element is $g = g_L(z) g_R^{-1}(\bar{z})$ and there are left and right moving currents. Therefore we now need two sets of oscillators, the left movers α_n^\pm, s_n , and the right movers $\tilde{\alpha}_n^\pm, \tilde{s}_n$. So, the direct product Hilbert space has a base labelled by $|p^-, p^+, s_0; \tilde{p}^-, \tilde{p}^+, \tilde{s}_0\rangle$ with $p^- p^+ = -r$ and $\tilde{p}^- \tilde{p}^+ = -\tilde{r}$ to insure that the currents obey the monodromy conditions in the physical sector. We now need to figure out if these are all independent labels or if they must be constrained by physical considerations.

For this purpose we recall that a possible modular invariant is the so called “diagonal invariant” that requires the same unitary representation labelled by the same j for both left and right movers. This may be understood as being related to the representation of the full group element $D^j(g) = D^j(g_L(z)) D^j(g_R^{-1}(\bar{z}))$ which requires the same j for both left and right movers. Therefore, we must demand $s_0 = \tilde{s}_0$.

In addition, we examine $g(z, \bar{z})$ in more detail. Keeping the order of operators, it may be written in the form

$$g = g_L(z) g_R^{-1}(\bar{z}) = \begin{pmatrix} u & a \\ -b & v \end{pmatrix} \quad (49)$$

with

$$\begin{aligned} u &= e^{\frac{-u_L + u_R}{k-2}} - e^{\frac{-u_L}{k-2}} (X_L^+ - X_R^+) X_R^- e^{\frac{-u_R}{k-2}} \\ v &= e^{\frac{u_L - u_R}{k-2}} + e^{\frac{-u_L}{k-2}} X_L^- (X_L^+ - X_R^+) e^{\frac{-u_R}{k-2}} \\ a &= e^{\frac{-u_L}{k-2}} (X_L^+ - X_R^+) e^{\frac{-u_R}{k-2}} \end{aligned} \quad (50)$$

$$b = - \left(X_L^- - X_R^- \right) e^{\frac{-u_L + u_R}{k-2}} + e^{\frac{-u_L}{k-2}} X_L^- \left(X_L^+ - X_R^+ \right) X_R^- e^{\frac{-u_R}{k-2}}$$

We see that g is not periodic under $\sigma \rightarrow \sigma + 2\pi n$ since there are logarithms in the expressions for every $X_{L,R}^\pm, u_{L,R}$. However, provided we impose $p^+ = -\tilde{p}^+$ on physical states (to cancel the non-periodic behavior in $X_L^+ - X_R^+$), we find that we can rewrite this monodromy in the form

$$g(ze^{i2\pi n}, \bar{z}e^{-i2\pi n}) = U\tilde{U}g(z, \bar{z})\tilde{U}^{-1}U^{-1} \quad (51)$$

$$U\tilde{U} = e^{-ip^+p^-2\pi n} e^{-is_0^2 2\pi n} e^{i\tilde{p}^+ \tilde{p}^- 2\pi n} e^{i\tilde{s}_0^2 2\pi n}$$

where p^+, s_0 are operators which do not commute with q^-, u_0 , and similarly for right movers (note that we have never introduced a canonical conjugate to p^- (or \tilde{p}^-)). To insure that the matrix elements of the overall g are consistent with monodromy in the physical sector it is sufficient to impose the conditions

$$2p^+p^- + 2s_0^2 - 2\tilde{p}^+\tilde{p}^- - 2\tilde{s}_0^2 = 2m \quad (52)$$

where m is an integer. Since we have already seen that $s_0 = \tilde{s}_0$ we find that this condition reduces to $r - \tilde{r} = m$, and does not impose any additional constraints on r, \tilde{r} . Furthermore, for a closed string we should also have $L_0 - \tilde{L}_0 = 0$ on the physical states. According to the mass shell condition (42) this requires $r - l = \tilde{r} - \tilde{l}$.

So, modular invariant physical closed string states are labelled at the base as follows

$$|p^-, p^+, s_0(n) \rangle \times |\tilde{p}^-, -p^+, s_0(n) \rangle \quad (53)$$

where the following restrictions are imposed

$$\begin{aligned} \tilde{p}^+ &= -p^+, & \tilde{s}_0 &= s_0 = s_0(n) \\ s_0(n) &\equiv [(k-2)(n+a) - 1/4]^{1/2} \end{aligned} \quad (54)$$

$$n = 0, 1, 2, \dots \quad (55)$$

Then the on-shell physical states automatically satisfy the periodicity conditions.

8 Chiral Vertex operators

In string theory there is a vertex operator corresponding to every physical state. In the usual algebraic approach one would try to construct a chiral vertex operator $V_{jm}(z)$ corresponding to the “tachyon” state $|jm\rangle$. In our case we have diagonalized $J_0^0 + J_0^1 \rightarrow p^+$ instead of $J_0^0 \rightarrow m$. Hence, our vertex operator is labelled as $V_{jp^+}(z)$. In this basis, we need a vertex operator with the following operator product properties

$$J^a(z) V_{jp^+}(w) \sim \frac{1}{z-w} [t^a V_{jp^+}(w)] \quad (56)$$

where t^a is a Hermitean generator of $SL(2, \mathbb{R})$ that acts on the p^+ label in the unitary principal series labelled by $j = -1/2 + is$. This representation is given by

$$\begin{aligned} (t^0 + t^1) V_{jp^+}(w) &= p^+ V_{jp^+}(w) \\ t^2 V_{jp^+}(w) &= \left[\frac{1}{2} \{p^+, i\partial_{p^+}\} + s \right] V_{jp^+}(w) \\ (t^0 - t^1) V_{jp^+}(w) &= [i\partial_{p^+} p^+ i\partial_{p^+} + 2si\partial_{p^+}] V_{jp^+}(w) \end{aligned} \quad (57)$$

A vertex operator with these properties is given by

$$V_{jp^+}(w) = \exp(ip^+ X^-(w)) : \exp\left(\frac{1+2is}{k-2} u(w)\right) : \quad (58)$$

It is straightforward to verify that it has the correct operator product properties with the currents in (24). This product of exponentials has a rather simple structure which can be readily manipulated in the computation of corellation functions.

There is a $\ln z$ term in the first exponential

$$\exp(p^+ \alpha_0^- \ln w) = w^{p^+ \alpha_0^-}$$

The monodromy condition requires that the power $p^+ \alpha_0^- = \text{integer}$, in agreement with the previous approach.

In the construction of corellation functions in other WZW models it has been found that there are screening charges that play an important role. In the $SL(2, \mathbb{R})$ case we found the following two screening currents (note that $S_1 \sim \partial X^+$)

$$\begin{aligned} S_1(z) &= \frac{P^+(z)}{k-2} : \exp\left(\frac{2u(z)}{k-2}\right) : \\ S_2(z) &= (P^+(z))^{k-2} : \exp(2u(z)) : \end{aligned} \quad (59)$$

Their operator products with the currents are

$$\begin{aligned} [J^0(z) + J^1(z)] S_1(w) &\sim 0 \\ J^2(z) S_1(w) &\sim 0 \\ [J^0(z) - J^1(z)] S_1(w) &\sim \partial_w \left(\frac{\exp[2u(w)/(k-2)]}{z-w} \right) \end{aligned} \quad (60)$$

and

$$\begin{aligned} [J^0(z) + J^1(z)] S_2(w) &\sim 0 \\ J^2(z) S_2(w) &\sim 0 \\ [J^0(z) - J^1(z)] S_2(w) &\sim \partial_w \left(\frac{(P^+(w))^{k-3} e^{2u(w)}}{z-w} \right) \end{aligned} \quad (61)$$

Therefore, the zero modes of these screening currents

$$Q_{1,2} \equiv \frac{1}{2\pi i} \oint dz S_{1,2}(z) \quad (62)$$

are the screening charges that commute with all the currents. These play an important role in the construction of correlation functions. We hope to discuss these issues further in future work.

9 Comments

We have shown that a unitary string theory in $SL(2, \mathbb{R})$ curved spacetime can be constructed and its spectrum solved exactly. A crucial ingredient was an additional zero mode whose presence introduces many new technical and physical features. This should provide a lesson for other more involved models.

Attention was drawn to some new technical points. The first is that currents are allowed to contain logarithmic cuts provided monodromy conditions are applied on the physical states. The second is a new representation of the currents in terms of free bosons that render the theory completely solvable. Vertex operators and screening currents that should be useful in computations were also suggested. These features have obvious generalizations to higher dimensions as well as to gauged WZW models (coset models).

We have also shown that the free boson methods permit a more general representation of $SL(2, \mathbb{R})$ current algebra when the extra degrees of freedom L'_n are introduced. These were absent in the WZW model, but they may be present in more general models.

As emphasized in the introduction, the main purpose for the present exercise is to develop the appropriate methods to study string theory during the early universe and to understand the impact of string theory on the symmetries and matter content observed at accelerator energies. For this purpose the current methods must be generalized to heterotic strings such as those described in [18]. Methods used for other special models of curved spacetimes may also be helpful [19].

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